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MORE EXAMPLES OF BICROSSPRODUCT AND DOUBLE CROSS PRODUCT HOPF ALGEBRAS

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ABSTRACT
We construct examples of bicrossproducts and double cross products of quantum groups $\check{A}(R)$ associated to general matrix solutions R of the Quantum Yang–Baxter Equations. We also describe iterated double cross products of quantum groups. In the course of constructing $\check{A}(R)$ we are led to introduce a suitable notion of mutually dual Hopf algebras and a dual quantum group $\check{U}(R)$.

Introduction

Hopf algebra bicrossproducts $H_1 \bowtie H_2$ for general Hopf algebras H_1, H_2 were introduced in [M1, Section 3.1] in connection with the author's approach to quantum mechanics combined with gravity [M]. The case when H_1 is cocommutative and H_2 is commutative was also introduced previously and independently in [Sing] in connection with a general theory of extensions. In this case a pair of Hopf algebras (H_1, H_2) is called *Abelian* [Sing]. This Abelian case was also used by [Tak] to show that certain Hopf algebras of Taft and Wilson were of this bicrossproduct (or "bismashproduct") form. In the present paper we give further examples of the non-Abelian theory of [M1], as well as examples of related double cross products. The theory is recalled in Section 1. A technical notion of *weak Hopf algebras* with *weak*

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antipodes is also introduced in Section 1 and is based on a notion of dually paired Hopf algebras. These constructions provide the abstract setting for our examples.

Section 2 constructs the class of Hopf algebras or weak Hopf algebras H_1, H_2 which will form the basis of our examples. They are motivated by constructions in physics coming under the heading *quantum groups* [Dri], [FRT], [M2]. In this literature there is a well-known construction of bialgebras $A(R)$ from matrices R obeying a certain matrix equation, the *Quantum Yang–Baxter Equation* (*QYBE*) [FRT]. We use the results of Section 1 to obtain instead actual Hopf algebras or weak Hopf algebras $\check{A}(R)$. The construction works for general sufficiently regular R. We illustrate the construction on a non-trivial example: The tensor algebra on matrices, $TM_n(k)$, is a self-dual weak Hopf algebra of this type. Unlike the quantum groups in the literature, this example is not associated to quantizations of Lie groups or Lie algebras.

In Section 3 we are finally able to construct examples of bicrossproducts and double cross products based on the weak Hopf algebras of Section 2. The double cross products focus on examples motivated by a finite dimensional construction of Drinfeld arising again from physics, the quantum double [Dri]. They take a very simple form in terms of "matrix multiplication", fully justifying the use of the terms *quantum matrices* for $A(R)$ and *quantum* (*matrix*) *groups* for $\check{A}(R)$. We also construct an iterated sequence of double cross products that are not of quantum double type. This sequence makes an unexpected connection with the parametrized Quantum Yang–Baxter Equation.

In the concluding remarks we show briefly how to include "cocycles" (χ, ψ) in the non-Abelian bicrossproduct theory. They can be interpreted as elements of some kind of "non-Abelian cohomology" $H^3(H_1, H_2)$ as a non-Abelian version of [Sing]. The search for examples of these cocycle bicrossproducts is therefore an interesting direction for further work.

For coproducts Δ, right actions and left coactions we shall frequently adopt the notations $\Delta(h) = h_{(1)} \otimes h_{(2)}$, $a \otimes h \mapsto a \triangleleft h$, $h \mapsto h^{(\bar{1})} \otimes h^{(\bar{2})}$ etc. Summations here are left understood. H^{op} denotes H with the opposite algebra structure, while H^{opc} denotes H with the opposite coalgebra structure. "Hopf algebra" means with antipode. Also, elements of H_1 will be denoted by $h, g, f, \ldots$ and elements of H_2 by $a, b, c, \ldots$. Thus $\varepsilon(h)$ means $\varepsilon_{H_1}(h)$ etc. k denotes an arbitrary commutative ground field. $M_n(k)$ denotes $n \times n$ matrices with values in k. If u is a matrix, u^i_j denotes the entry at row i, column j. τ denotes the twist map on various spaces and id the linear identity.

1. General constructions

Let H_1 and H_2 be Hopf algebras over k. We briefly recall the construction of bicrossproducts and double cross products from [M1, Section 3]. After this we give some new constructions needed in the paper.

PROPOSITION 1.1 [M1, Theorem 3.3]. *Let H_2 be a right H_1-module algebra by α, H_1 a left H_2-comodule coalgebra by β obeying the compatibility conditions*

$$\varepsilon(a \triangleleft h) = \varepsilon(a)\varepsilon(h), \quad \Delta(a \triangleleft h) = (a_{(1)} \triangleleft h_{(1)})h_{(2)}{}^{(\bar{1})} \otimes a_{(2)} \triangleleft h_{(2)}{}^{(\bar{2})} \qquad \text{(A)}$$

$$\beta(1) = 1 \otimes 1, \quad \beta(hg) = (h^{(\bar{1})} \triangleleft g_{(1)})g_{(2)}{}^{(\bar{1})} \otimes h^{(\bar{2})}g_{(2)}{}^{(\bar{2})} \qquad \text{(B)}$$

$$h_{(1)}{}^{(\bar{1})}(a \triangleleft h_{(2)}) \otimes h_{(1)}{}^{(\bar{2})} = (a \triangleleft h_{(1)})h_{(2)}{}^{(\bar{1})} \otimes h_{(2)}{}^{(\bar{2})} \qquad \text{(C)}$$

for all $h, g \in H_1$, $a \in H_2$. Then the linear space $H_1 \otimes H_2$ endowed with the smash product algebra by α and the smash coproduct coalgebra by β is a Hopf algebra. It is denoted $H_1 {}^{\beta}\!\bowtie_{\alpha} H_2$, the bicrossproduct (or "bismashproduct") Hopf algebra.

In our notation, the lowered α denotes an action while the raised β denotes a coaction. If H_1, H_2 are merely bialgebras the bicrossproduct makes sense as a bialgebra. Bicrossproducts are characterized abstractly in [M1] in terms of the universal properties of smash products and smash coproducts. They are also extensions of (H_1, H_2) in the sense of [Sing]. The bicrossproduct data $(H_1, H_2, \alpha, \beta)$ can also be used to construct a Hopf algebra structure on the convolution algebra $\mathrm{Hom}(H_1, H_2)$. In the finite dimensional case, this can be expressed as a construction on $H_1^* \otimes H_2$. We denote its dual Hopf algebra by $H_1 {}_{\beta^*}\!\bowtie_{\alpha^*} H_2^*$ where α^*, β^* are now actions obtained by suitably dualizing α, β [M1, Proposition 3.13]. This is an example of a double cross product:

PROPOSITION 1.2 [M1, Section 3.2]. *Let H_1, H_2 be Hopf algebras over k. Let α be a left H_1-module coalgebra structure on H_2, and β a right H_2-module coalgebra structure on H_1 such that*

$$h \triangleright 1_{H_2} = \varepsilon_{H_1}(h)1_{H_2}, \quad h \triangleright (ab) = (h_{(1)} \triangleright a_{(1)})((h_{(2)} \triangleleft a_{(2)}) \triangleright b) \qquad \text{(A)}$$

$$1_{H_1} \triangleleft a = 1_{H_1}\varepsilon_{H_2}(a), \quad (hg) \triangleleft a = (h \triangleleft (g_{(1)} \triangleright a_{(1)}))(g_{(2)} \triangleleft a_{(2)}) \qquad \text{(B)}$$

$$h_{(1)} \triangleleft a_{(1)} \otimes h_{(2)} \triangleright a_{(2)} = h_{(2)} \triangleleft a_{(2)} \otimes h_{(1)} \triangleright a_{(1)} \qquad \text{(C)}$$

for all $h, g \in H_1$, $a, b \in H_2$. Here $\alpha_h(a) = h \triangleright a$ and $\beta_a(h) = h \triangleleft a$ denote the left *and right module coalgebra structures. Then the linear space* $H_1 \otimes H_2$ *has a Hopf algebra structure denoted* $H_1 {}_\beta\bowtie_\alpha H_2$, *the* double cross product.

Explicitly, the Hopf algebra structure is

$$(h \otimes a)(g \otimes b) = (h_{(2)} \triangleleft b_{(2)})g \otimes a(h_{(1)} \triangleright b_{(1)}), \qquad 1 = 1 \otimes 1,$$

$$\Delta(h \otimes a) = h_{(1)} \otimes a_{(1)} \otimes h_{(2)} \otimes a_{(2)}, \quad \varepsilon(h \otimes a) = \varepsilon(h)\varepsilon(a),$$

$$S(h \otimes a) = (Sh \otimes 1)(1 \otimes Sa).$$

We also say that $(H_1, H_2, \alpha, \beta)$ obeying these equations (A)–(C) of Proposition 1.2 constitute a *matched pair*. If H_1, H_2 are merely bialgebras, then so is $H_1 {}_\beta\bowtie_\alpha H_2$. The double cross product has a characteristic factorization property [M1]. In particular it contains both H_1, H_2 as subhopf algebras, and the map $H_2 \otimes H_1 \to H_1 {}_\beta\bowtie_\alpha H_2 : a \otimes h \mapsto ah$ is bijective.

If H is any (not necessarily finite dimensional) Hopf algebra with bijective antipode, we showed in [M1, Section 4] how to construct a bicrossproduct $H^{\mathrm{co}_\iota} \bowtie_{\mathrm{ad}_\iota} H^{\mathrm{op}}$. Here $\alpha = \mathrm{ad}_\iota$, $\beta = \mathrm{co}_\iota$ are respectively the adjoint action and adjoint coaction induced by the linear identity map $\iota : H \to H^{\mathrm{op}}$. According to the above remarks, if H is finite dimensional we also have a related double cross product $H_{\mathrm{co}_\iota^*} \bowtie_{\mathrm{ad}_\iota^*} H^{*\mathrm{opc}}$. Here $H^{*\mathrm{opc}}$ is the dual Hopf algebra with opposite coproduct. We identified this in [M1, Section 4] as $D(H)$, the *quantum double* Hopf algebra of Drinfeld. This quantum double $D(H)$ is an important example arising in mathematical physics [Dri] but so far is limited to H finite dimensional. This completes the review of [M1].

The remainder of this section consists of new results that will be needed for the examples of the paper. The results include a novel notion of dual bialgebras or Hopf algebras in the form of dual pairings (needed in Section 2), and an application of this to construct infinite dimensional double cross products analogous to $D(H)$ (needed in Section 3).

DEFINITION 1.3. Two bialgebras or Hopf algebras H_1, H_2 are said to be *dually paired* if there is a bilinear map $\langle\ ,\ \rangle : H_1 \otimes H_2 \to k$ such that

$$\langle hg, a\rangle = \langle h \otimes g, \Delta a\rangle, \quad \langle 1, a\rangle = \varepsilon(a), \quad \langle \Delta h, a \otimes b\rangle = \langle h, ab\rangle, \quad \langle h, 1\rangle = \varepsilon(h)$$

and in the Hopf case $\langle Sh, a\rangle = \langle h, Sa\rangle$ for all $h, g \in H_1$, $a, b \in H_2$. They are said to be *mutually dual* if the pairing is non-degenerate. This means that $\langle h, a\rangle = 0$ for all $a \in H_2$ implies $h = 0$, and $\langle h, a\rangle = 0$ for all $h \in H_1$ implies $a = 0$.

PROPOSITION 1.4. *Let H_1, H_2 be dually paired bialgebras (or Hopf algebras). Let*

$$K_1=\{h\in H_1 \mid \langle h,a\rangle=0\ \forall a\in H_2\},\quad K_2=\{a\in H_2\mid \langle h,a\rangle=0\ \forall h\in H_1\}$$

(the kernels of the pairing). Then K_1, K_2 are bi-ideals and H_1/K_1 and H_2/K_2 are mutually dual bialgebras (or Hopf algebras).

PROOF. The proof is elementary and is omitted.

Note that if H_1, H_2 are mutually dual, the maps $h\mapsto\langle h,\ \rangle$ and $a\mapsto\langle\ ,a\rangle$ each define canonical algebra inclusions $H_1\subseteq H_2^*$ and $H_2\subseteq H_1^*$. In the finite dimensional case this implies that $H_1=H_2^*$. So the notion of mutually dual Hopf algebras is a generalization of finite dimensional duality.

DEFINITION 1.5. Dually paired bialgebras H_1, H_2 are said to possess *weak antipodes* if the canonical maps $H_1\xrightarrow{j_{H_1}}H_2^*$ and $H_2\xrightarrow{j_{H_2}}H_1^*$ defined by the pairing are invertible in the convolution algebras $\mathrm{Hom}(H_1,H_2^*)$ and $\mathrm{Hom}(H_2,H_1^*)$ respectively. These inverses, the weak antipodes, will be denoted by S_{H_1}, S_{H_2} (or simply S).

LEMMA 1.6. *Let H_1, H_2 be dually paired bialgebras.* (i) *If one of the weak antipodes exists then by dualizing, so does the other and is necessarily compatible in the sense $(Sh)(a)=(Sa)(h)$ for all $h\in H_1$, $a\in H_2$.* (ii) *If weak antipodes exist, they obey*

$$S(hg)=(Sg)(Sh),\ S1=1,\ (Sh)(ab)=(Sh_{(2)})(a)(Sh_{(1)})(b),\ (Sh)(1)=\langle h,1\rangle$$

for all $h,g\in H_1$, $a,b\in H_2$, and similarly for the weak antipode on H_2. (iii) *If H_1, H_2 are mutually dual with weak antipodes, H_1 has an ordinary antipode iff $S(H_1)$ is contained in the image of $H_1\subseteq H_2^*$, in which case $(Sh)(a)=\langle Sh,a\rangle$. Similarly for H_2. For example, in the finite dimensional case, mutually dual bialgebras with weak antipodes are just ordinary dual Hopf algebras $H_1=H_2^*$.*

PROOF. (i) If $S_{H_1}:H_1\to H_2^*$ is a weak antipode for H_1, then define S_{H_2} to be $S_{H_1}^*$ precomposed with the canonical inclusion $H_2\subseteq H_2^{**}$. (ii) The proof is similar to the usual proofs for an ordinary antipode [Swe]. Use associativity in the convolution algebra $\mathrm{Hom}(H_1,H_2^*\otimes H_2^*)$ etc. (iii) is elementary. These observations justify the term "weak antipode".

Dually paired bialgebras with weak antipodes are therefore called *weak Hopf* *algebras*. In this context we are now able to state the generalized quantum double construction.

THEOREM 1.7. *Let H_1, H_2 be dually paired bialgebras with weak antipodes. Then $(H_1, H_2^{\rm op})$ is a matched pair. The actions are*

$$\alpha_h(\imath(a)) = \imath(a_{(2)})((S_{H_2}a_{(1)})j_{H_2}(a_{(3)}))(h), \quad \beta_{\imath(a)}(h) = h_{(2)}((S_{H_1}h_{(1)})j_{H_1}(h_{(3)}))(a)$$

$\forall\, h \in H_1$, $a \in H_2$. Here $\imath : H_2 \to H_2^{\rm op}$ is the linear identity. We call a matched pair of this form a matched pair of coadjoint type, *and $H_1\, {}_\beta\!\bowtie_\alpha H_2^{\rm op}$ the associated* double. *Explicitly, the structure of the double is*

$$(h \otimes \imath(a))(g \otimes \imath(b)) = (Sh_{(1)})(b_{(1)})h_{(2)}g \otimes \imath(a)\imath(b_{(2)})\langle h_{(3)}, b_{(3)}\rangle,$$

$$h, g \in H_1,\ a, b \in H_2.$$

If the weak antipodes descend to the quotients, we also obtain a double $H_1/K_1 \bowtie (H_2/K_2)^{\rm op}$.

PROOF. This follows from direct computation using Lemma 1.6 to check that the stated α, β are indeed module coalgebra structures and that axioms (A)–(C) in Proposition 1.2 hold. Note also that in the finite dimensional case $H_1 = H$, $H_2 = H^*$, the double is $H_\beta \bowtie_\alpha H^{*\rm op} \cong H_{\rm co^*_{\!\!\!\!\!\!\;}} \bowtie_{\rm ad^*} H^{*\rm opc} \cong D(H)$, the quantum double. The first isomorphism here is given by the antipode on H viewed as a linear map $S : H^{*\rm opc} \to H^{*\rm op}$.

2. Construction of $\check{A}(R)$

In this and the next section, the following notations will be useful. If $R \in M_n(k) \otimes M_n(k)$, we let R_{ab} denote R embedded in $M_n(k) \otimes M_n(k) \cdots \otimes M_n(k)$ in the a, b position and 1 elsewhere. If A is an algebra, $M_n(A) = M_n(k) \otimes A$ denotes $n \times n$ matrices with entries in A. If $u, v \in M_n(A)$, $uv \in M_n(A)$ is the usual product (i.e. matrix multiplication and multiplication in A). We shall need to work with tensor products separately in A and $M_n(k)$. We use $\otimes$ to denote tensor product in A, and $\hat{\otimes}$ for tensor product in $M_n(k)$. Thus $u \otimes v \in M_n(A \otimes A)$ (matrix multiplication in $M_n(k)$ and tensor product in A), while $u \hat{\otimes} v \in M_n(A) \hat{\otimes} M_n(A) = M_n(k) \otimes M_n(k) \otimes A$ (tensor product in $M_n(k)$, multiplication in A). Finally, we denote N-fold $\hat{\otimes}$ powers of $M_n(A)$ as $M_n^N(A)$. We denote the various embeddings $M_n(A) \hookrightarrow M_n^N(A)$ by $u \mapsto u_a = 1 \hat{\otimes} 1 \cdots \hat{\otimes} u \cdots \hat{\otimes} 1$ (u in the a'th position). Thus $u_1u_2 = (u \hat{\otimes} 1)(1 \hat{\otimes} u) = u \hat{\otimes} u$. This is generally not the same as $u_2u_1 = (1 \hat{\otimes} u)(u \hat{\otimes} 1)$, unless A is commutative.

The notation is motivated from mathematical physics and constitutes a useful "quantum matrix algebra".

The *Quantum Yang–Baxter Equation* (*QYBE*) for $R \in M_n^2(k)$ is the matrix equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \qquad \text{(QYBE)}$$

in $M_n^3(k)$. R is called invertible if it is invertible as a matrix. We shall also require a further regularity condition below. In this section, we associate to each invertible regular solution R of the QYBE a pair of Hopf algebras $\check{A}(R)$ and $\check{U}(R)$. The first step of the construction, to obtain a bialgebra $A(R)$, is well known; see [FRT] where also the relations that we use in $\check{U}(R)$ below were identified in their study of the dual of $A(R)$. The rest of the general construction is new but recovers well known "quantum groups" such as $\mathrm{SL}_q(2)$ and $U_q(\mathrm{sl}_2)$ for particular choice of R; cf. [FRT]. Previously, the additional relations (such as "quantum determinants") needed to obtain Hopf algebras from $A(R)$ were known only for these particular choices of R, being introduced in an ad-hoc manner.

The bialgebra $A(R)$ is defined as the free non-commutative algebra over k generated by 1 and n^2 indeterminates $u \in M_n(A(R))$ modulo the relations

$$Ru_1u_2 = u_2u_1R.$$

Here and throughout the paper we regard these n^2 generators together as a matrix of generators. In this notation, $A(R)$ becomes a bialgebra with

$$\Delta u = u \otimes u, \qquad \varepsilon u = 1 \in M_n(k),$$

extended as algebra maps to $A(R)$. $A(R)$ is called the bialgebra of *quantum matrices* associated to R. Note that $u \otimes u$ involves matrix multiplication so that for $n > 1$ the bialgebra is generally both non-commutative and non-cocommutative.

Motivated by [FRT], we also define a bialgebra $\tilde{U}(R)$ as the algebra generated by 1 and $2n^2$ indeterminates $l^+, l^- \in M_n(\tilde{U}(R))$ modulo relations

$$l_1^+l_2^+R = Rl_2^+l_1^+, \quad l_1^-l_2^-R = Rl_2^-l_1^-, \quad l_1^-l_2^+R = Rl_2^+l_1^-.$$

It is also a bialgebra with

$$\Delta l^+ = l^+ \otimes l^+, \quad \Delta l^- = l^- \otimes l^-, \quad \varepsilon l^{\pm} = 1.$$

We shall see in Section 3 that $\tilde{U}(R)$ is in fact a double cross product of two copies of $A(R)^{\mathrm{op}}$.

PROPOSITION 2.1. *Let R be an invertible solution of the QYBE. The bialgebras $A(R)$ and $\tilde{U}(R)$ are dually paired by*

$$\langle u_1, l_2^+\rangle = R, \qquad \langle u_1, l_2^-\rangle = \tau(R^{-1})$$

(cf. [FRT]). *Here τ denotes the twist map on $M_n^2(k)$. Consequently from Proposition* 1.4 *we have mutually dual bialgebras*

$$\check{A}(R) = A(R)/K_1, \qquad \check{U}(R) = \tilde{U}(R)/K_2.$$

We call $\check{A}(R)$ the quantum matrix group *associated to R and $\check{U}(R)$ its dual.*

PROOF. It can easily be checked that the pairing is well defined when extended as an algebra map in each input and in such a way that the pairing holds. Thus, for example,

$$R_{12}\langle u_1u_2, l_3^+\rangle = R_{12}\langle u_1 \otimes u_2, \Delta l_3^+\rangle = R_{12}\langle u_1 \otimes u_2, l_3^+ \otimes l_3^+\rangle = R_{12}R_{13}R_{23}$$

while $\langle u_2u_1, l_3^+\rangle R_{12} = \langle u_2 \otimes u_1, l_3^+ \otimes l_3^+\rangle R_{12} = R_{23}R_{13}R_{12}$. This observation was effectively first made in [FRT] who identified the relations that we have used in $\tilde{U}(R)$ as among the relations satisfied in a dual of $A(R)$. The proposition then follows as a corollary of Proposition 1.4.

PROPOSITION 2.2. *Let R be an invertible solution of the QYBE. $A(R)$, $\tilde{U}(R)$ constructed above, possess weak antipodes. They are defined by*

$$(Su_1)(l_2^+) = (Sl_2^+)(u_1) = R^{-1}, \qquad (Su_1)(l_2^-) = (Sl_2^-)(u_1) = \tau(R).$$

If these weak antipodes descend to ordinary antipodes on $\check{A}(R)$, $\check{U}(R)$, we say that R is regular.

PROOF. Here the S shown are extended as antialgebra maps to all of $A(R)$ and $\tilde{U}(R)$. It is easy to check, as for Proposition 2.1, that extension as algebra maps is well defined. Let $j: A(R) \to \tilde{U}(R)^*$ be the canonical map. Then,

$$\begin{aligned} j(u_1)(Su_1)(l_2^+l_3^+\cdots l_N^+) &= (j(u_1)\otimes Su_1)(\Delta l_2^+\cdots l_N^+) \\ &= \langle u_1, l_2^+l_3^+\cdots l_N^+\rangle(Su_1)(l_2^+l_3^+\cdots l_N^+) \\ &= R_{12}R_{13}\cdots R_{1N}R_{1N}^{-1}\cdots R_{12}^{-1} = 1_{M_n^N(k)} = \varepsilon(u_1)\langle 1, l_2^+l_3^+\cdots l_N^+\rangle. \end{aligned}$$

Similarly for the other facts. Note that what is required of the antipode on $A(R)$ is $uSu = (Su)u = 1 \in M_n(A(R))$. Thus Su is precisely the inverse, u^{-1}, for the product in $M_n(A(R))$. Formally, $A(R)$ (and hence $\check{A}(R)$) have such an inverse, given by the formal power-series $Su = \Sigma_{M=0}^{\infty}(1-u)^M$ (cf. the formal inverse in $M_n(k)$). By contrast, the weak antipode on $A(R)$ and its projection on $\check{A}(R)$ corresponds to an explicit formula for the inverse (cf. the formula for inversion of 2×2 matrices of unit determinant). The weak antipode on $\tilde{U}(R)$ is similar.

Also, it can be shown by similar computations that $\check{U}(R)$ is essentially quasitriangular in the sense of [Dri]. Indeed, this notion can be made precise in the context of dually paired bialgebras H_1, H_2 as the existence of an antialgebra and coalgebra map $\mathscr{R}: H_1 \to H_2$, invertible in the convolution algebra $\mathrm{Hom}(H_1, H_2)$ and such that $\Delta_1 a = \mathscr{R} * \Delta_2 a * \mathscr{R}^{-1}$ for all $a \in H_2$. Here $\Delta_1 a, \Delta_2 a: H_1 \to H_2$ are defined by $(\Delta_1 a)(h) = \langle a_{(1)}, h \rangle a_{(2)}$ and $(\Delta_2 a)(h) = a_{(1)} \langle a_{(2)}, h \rangle$. In the finite dimensional mutually dual case this is equivalent to H_2 quasitriangular. For the above example the required map $\mathscr{R}: A(R) \to \tilde{U}(R)$ is $\mathscr{R}(u) = l^+$. For sufficiently nice R this descends to the quotients $\check{A}(R)$ and $\check{U}(R)$ (so in this context we say that R is regular if the weak antipodes and $\mathscr{R}$ descend). An application of these results to physics is in [M2].

Example 2.3 ($R = 1 \otimes 1$). If R is the identity in $M_n^2(k)$, then $A(R) \cong k(M_n(k))$, the commutative algebra of polynomials in n^2 variables, and $\check{A}(R) = k$. Likewise $\check{U}(R) = k$. This R is regular.

Example 2.4 ($R = \tau$). If $n > 1$ and $R = \tau$, the twist map $k^n \otimes k^n \to k^n \otimes k^n$, then $A(R) = \check{A}(R) \cong TM_n(k)$, the free algebra generated by 1 and n^2 indeterminates u with no relations. We learn from the above that this is a bialgebra with $\Delta u = u \otimes u$ and, moreover, has a weak antipode. $\check{U}(R) \cong TM_n(k)$ also (and is quasitriangular). Let $\check{A}(R)$ and $\check{U}(R)$ be graded by the degree in the generators. Then the weak antipode on $\check{A}(R)$ is defined by

$$(Su)(a) = \langle u^N, a \rangle$$

for all $a \in \check{U}(R)$ homogeneous of degree N. Similarly for the weak antipode on $\check{U}(R)$.

Proof. τ implements the twist map on $M_n^2(k)$, so $\tau u_1 u_2 \tau^{-1} = u_2 u_1$. Hence the relations in $A(R)$ in this example are empty. As a matrix in $M_n^2(k)$, τ is symmetric under the twist on $M_n^2(k)$. Hence $l^+ - l^- \in K_2$. Let $U(R) = \tilde{U}(R)/\{l^+ - l^-\}$. Then both $A(R)$ and $U(R)$ are the free algebra on n^2 generators, $u = \{u^i_j\}^n_{i,j=1}$ and $l = \{l^k_l\}^n_{k,l=1}$ respectively. We have to show that the pairing between $A(R)$ and $U(R)$ is now non-degenerate. Explicitly writing the matrix entries, the pairing is

$$\langle u^{i_1}_{j_1} u^{i_2}_{j_2} \cdots u^{i_M}_{j_M}, l^{k_1}_{l_1} l^{k_2}_{l_2} \cdots l^{k_N}_{l_N} \rangle$$

$$= \begin{cases} \delta^{i_M}_{l_1} \cdots \delta^{i_1}_{l_M} \delta^{k_1}_{l_{M+1}} \cdots \delta^{k_{N-M}}_{l_N} \delta^{k_{N-M+1}}_{j_M} \cdots \delta^{k_N}_{j_1}, & M < N \\ \delta^{i_M}_{l_1} \cdots \delta^{i_1}_{l_N} \delta^{k_1}_{j_M} \cdots \delta^{k_N}_{j_1}, & M = N \\ \delta^{i_M}_{l_1} \cdots \delta^{i_{M-N+1}}_{l_N} \delta^{i_{M-N}}_{j_M} \cdots \delta^{i_1}_{j_{N+1}} \delta^{k_1}_{j_N} \cdots \delta^{k_N}_{j_1}, & M > N \end{cases}$$

where δ is the identity matrix. Using this, it is easy to see that for any given N the elements

$$e_{(N)}{}^{j'_1 \cdots j'_N}_{i'_1 \cdots i'_N} = l^{k_1}_{i'_N} l^{k_2}_{i'_{N-1}} \cdots l^{k_N}_{i'_1} l^{j'_N}_{k_1} \cdots l^{j'_1}_{k_N}, \qquad i'_1 \notin \{k_1, k_1, \cdots, k_N\}$$

(choose any such k_i for given i', j') have zero pairing with all $h \in A(R)$ of degree $< N$ and

$$\langle u^{i_1}_{j_1} \cdots u^{i_N}_{j_N}, e_{(N)}{}^{j'_1 \cdots j'_N}_{i'_1 \cdots i'_N} \rangle = \delta^{i_1}_{i'_1} \cdots \delta^{i_N}_{i'_N} \delta^{j'_1}_{j_1} \cdots \delta^{j'_N}_{j_N}$$

on the subspace of $A(R)$ homogeneous of degree N. Now suppose $c \in K_1$ has finite top degree, say $c = c_N + c_{N-1} + \cdots + c_0$ (with c_i homogeneous of degree i). By choosing the vectors $e_{(N)}$, which pair only with c_N, we conclude $\langle c_N, e_{(N)} \rangle = 0$. But $e_{(N)}$ are a basis for the dual space of the subspace of elements homogeneous degree N. Hence $c_N = 0$. Hence by induction there are no non-zero elements of finite top degree. Similarly K_2 is trivial and $\check{U}(R) = U(R)$. The definition of S in Proposition 2.2 is $(Su^i_j)(l^{k_1}_{l_1} \cdots l^{k_N}_{l_N}) = \delta^i_{l_N} \delta^{k_1}_j \delta^{k_2}_{l_1} \cdots k^{k_N}_{l_{N-1}}$. This coincides with $\langle u^N, l^{k_1}_{l_1} \cdots l^{k_N}_{l_N} \rangle$. It is also easy to compute similarly that $\langle u^{N+1}, a \rangle = \langle 1, a \rangle$ for all $a \in \check{U}(R)$ homogeneous of degree N, so that Su should indeed be thought of weakly as u^{-1} under matrix multiplication.

Between these extremes there are many solutions R of the QYBE. Many of these, see e.g. [Resh], involve a parameter $q \in k^*$, and are regular. Moreover, the corresponding antipode is bijective. For example,

$$R = q^{-1/2} \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{bmatrix} \in M_2(M_2(k)) = M_2^2(k)$$

(we suppose q has a square root in k) leads to the Hopf algebra $SL_q(2)$ with antipode explicitly given by

$$Su = \begin{pmatrix} u^2_2 & -qu^1_2 \\ -q^{-1}u^2_1 & u^1_1 \end{pmatrix},$$

which is bijective. For these R, as $q \to 1$ the associated Hopf algebras reduce to the commutative Hopf algebras generated by matrix elements of the defining representations of matrix groups. Hence the name "q-analog matrix group" or "quantum matrix group" for $\check{A}(R)$.

3. Bicrossproducts and double cross products $A(R)$ and $\check{A}(R)$

In the preceding section we have obtained infinite dimensional Hopf algebras or weak Hopf algebras associated to sufficiently regular solutions R of the Quantum Yang–Baxter Equations. Since these are generally non-commutative and non-cocommutative they provide non-trivial examples of the class of bicrossproducts of the form $H^{\mathrm{co}_\iota} \bowtie_{\mathrm{ad}_\iota} H^{\mathrm{op}}$ described in Section 1:

Proposition 3.1. *Let R be a regular invertible solution of the QYBE so that $\check{A}(R)$ is a Hopf algebra. We suppose that the antipode S is bijective. Let $\check{A}(R)^{\mathrm{op}}$ denote $\check{A}(R)$ with the opposite algebra. We denote its generators $v \in M_n(\check{A}(R)^{\mathrm{op}})$ and the linear identity map $\check{A}(R) \to \check{A}(R)^{\mathrm{op}}$ by ι. Then*

$$\alpha_{u_2}(v_1) = \iota(u_2) v_1 \iota(S^{-1}u_2), \qquad \beta(u) = \iota(u_{(1)})\iota(S^{-1}u_{(3)}) \otimes u_{(2)}$$

define a non-Abelian bicrossproduct $\check{A}(R)^{\beta} \bowtie_{\alpha} \check{A}(R)^{\mathrm{op}}$. Here α, β are ad_ι, co_ι *induced by ι and $u_{(1)} \otimes u_{(2)} \otimes u_{(3)} = u \otimes u \otimes u$.*

Proof. Explicit formulae for ad_ι, co_ι are given for arbitrary H with bijective antipode in [M1]. The above are then immediate for $H = \check{A}(R)$. Explicitly in terms of the matrix entries, α, β are $\alpha_{u^i_j}(v^k_l) = \Sigma_m \iota(u^i_m) v^{k}{}_{l}\, \iota(S^{-1}u^m_j)$ and $\beta(u^i_j) = \Sigma_{m_1,m_2} \iota(u^i_{m_1})\iota(S^{-1}u^{m_2}_j) \otimes u^{m_1}_{m_2}$.

This example demonstrates the beginnings of a "non-commutative algebra-valued" or "quantum" matrix calculus. The action α is like the adjoint action of invertible matrices, but intrinsically "non-commutative" in that it is trivial if $\check{A}(R)$ is commutative. The following related double crossproducts develop this similarity with matrix multiplication much further.

In the remainder of this section we are going to give examples of infinite dimensional double crossproducts involving $A(R)$ and $\check{A}(R)$, with the finite dimensional quantum double as a model. Firstly however, we find that $\tilde{U}(R)$, introduced in Section 2, is a double cross product.

Theorem 3.2. *Let R be an invertible solution of the QYBE and $A(R)$ the associated bialgebra of Section 2. Then $(A(R), A(R))$ are a matched pair of bialgebras and $A(R)_{\beta} \bowtie_{\alpha} A(R) \cong \tilde{U}(R)^{\mathrm{op}}$. Likewise, $(A(R)^{\mathrm{op}}, A(R)^{\mathrm{op}})$ are a matched pair and $A(R)^{\mathrm{op}}\,{}_{\beta}\bowtie_{\alpha} A(R)^{\mathrm{op}} \cong \tilde{U}(R)$. Explicitly, let w and v denote the* *matrix generators of the respective copies of $A(R)^{\rm op}$. Then the actions for this case are*

$$\alpha_{w_2}(v_1)=R^{-1}v_1R,\qquad \beta_{v_1}(w_2)=R^{-1}w_2R$$

extended as a matched pair.

PROOF. The cases of $A(R)$ and $A(R)^{\rm op}$ are strictly analogous (as $A(R)^{\rm op}=A(R^{-1})=A(\tau(R))$). We give the proof explicitly for the latter. $A(R)^{\rm op}$ is the free algebra on n^2 generators, say w with relations $w_1w_2R=Rw_2w_1$. We must check that α,β respect such relations. (1) First we check that α extends as an action of $A(R)^{\rm op}$,

$$\begin{aligned} w_2w_3R_{23}\triangleright v_1 &= w_2\triangleright R_{13}^{-1}v_1R_{13}R_{23}=R_{13}^{-1}R_{12}^{-1}v_1R_{12}R_{13}R_{23}\\ &=R_{13}^{-1}R_{12}^{-1}v_1R_{23}R_{13}R_{12}=R_{13}^{-1}R_{12}^{-1}R_{23}v_1R_{13}R_{12}\\ &=R_{23}R_{12}^{-1}R_{13}^{-1}v_1R_{13}R_{12}\\ &=R_{23}w_3\triangleright R_{12}^{-1}v_1R_{12}=R_{23}w_3w_2\triangleright v_1\end{aligned}$$

as required. We used the QYBE. Similarly, for β. (2) Next we check that α and β both extend according to (A)–(B) in Proposition 1.2, i.e. according to

$$w_3\triangleright v_1v_2=(w_3\triangleright v_1)((w_3\triangleleft v_1)\triangleright v_2),\qquad w_2w_3\triangleleft v_1=(w_2\triangleleft(w_3\triangleright v_1))(w_3\triangleleft v_1).$$

Thus using the QYBE and the relations for $A(R)^{\rm op}$,

$$\begin{aligned} w_3\triangleright v_1v_2R_{12} &=(w_3\triangleright v_1)((w_3\triangleleft v_1)\triangleright v_2)R_{12}=R_{13}^{-1}v_1R_{13}(R_{13}^{-1}w_3R_{13}\triangleright v_2)R_{12}\\ &=R_{13}^{-1}v_1R_{13}(R_{13}^{-1}w_3\triangleright v_2)R_{13}R_{12}=R_{13}^{-1}v_1R_{23}^{-1}v_2R_{23}R_{13}R_{12}\\ &=R_{13}^{-1}R_{23}^{-1}v_1v_2R_{12}R_{13}R_{23}=R_{13}^{-1}R_{23}^{-1}R_{12}v_2v_1R_{13}R_{23}\\ &=R_{12}R_{23}^{-1}R_{13}^{-1}v_2v_1R_{13}R_{23}=w_3\triangleright R_{12}v_2v_1\end{aligned}$$

as required. Hence also for any finite product $v_2\cdots v_N$,

$$w_1\triangleright v_2\cdots v_N=(w_1\triangleright v_2\cdots v_{N-1})(w_1\triangleleft v_2\cdots v_{N-1})\triangleright v_N$$

is well-defined by induction on N and by part (1) for β an action. Similarly, β extends to an action on $A(R)^{\rm op}$. (3) These actions are in fact module coalgebra structures,

$$(\Delta w_2)\triangleleft(v_1\otimes v_1)=(w_2\otimes w_2)\triangleleft(v_1\otimes v_1)=R_{12}^{-1}w_2R_{12}\otimes R_{12}^{-1}w_2R_{12}=\Delta(w_2\triangleleft v_1).$$

(4) Finally, condition (C) is satisfied: the left hand side is $w_2\triangleleft v_1\otimes w_2\triangleright v_1=R_{12}^{-1}w_2R_{12}\otimes R_{12}^{-1}v_1R_{12}=R_{12}^{-1}w_2\otimes v_1R_{12}$ while the right hand side is

$\tau(w_2 \triangleright v_1 \otimes w_2 \triangleleft v_1) = \tau(R_{12}^{-1} v_1 R_{12} \otimes R_{12}^{-1} w_2 R_{12}) = \tau(R_{12}^{-1} v_1 \otimes w_2 R_{12})$ where τ is the twist map on $A(R)^{\mathrm{op}} \otimes A(R)^{\mathrm{op}}$. Similarly, one may check that (C) holds for higher products of generators. (4) Hence we have a matched pair. Computing the associated double cross product we find

$$(w_2 \otimes 1)(1 \otimes v_1) = \tau(w_2 \triangleright v_1 \otimes v_1 \triangleleft w_2) \stackrel{(C)}{=} w_2 \triangleleft v_1 \otimes w_2 \triangleright v_1$$

$$= R_{12}^{-1}(w_2 \otimes v_1) R_{12} = R_{12}^{-1}(1 \otimes v_1)(w_2 \otimes 1) R_{12}.$$

Writing $l^+ = (w \otimes 1)$, $l^- = (1 \otimes v)$ we have that $A(R)^{\mathrm{op}} \bowtie A(R)^{\mathrm{op}}$ is generated by l^+, l^- with the relations given for $\tilde{U}(R)$ in Section 2. Also, $\Delta(w \otimes 1) = (w \otimes 1) \otimes (w \otimes 1)$ etc. so that the coalgebra structure is also as for $\tilde{U}(R)$.

Proposition 3.3. *Let R be an invertible solution of the QYBE and $A(R)$, $\tilde{U}(R)$ as above. $(\tilde{U}(R), A(R)^{\mathrm{op}})$ is a matched pair. If the weak antipodes descend (e.g., if R is regular) then $(\check{U}(R), \check{A}(R)^{\mathrm{op}})$ is a matched pair. The actions are*

$$\alpha_{l_2^+}(v_1) = R^{-1} v_1 R, \qquad \alpha_{l_1^-}(v_2) = R v_2 R^{-1},$$

$$\beta_{v_1}(l_2^+) = R^{-1} l_2^+ R, \qquad \beta_{v_2}(l_1^-) = R l_1^- R^{-1}$$

Proof. This is an example of Theorem 1.7 (i.e. an infinite dimensional analog of the quantum double construction). The necessary weak antipodes were found in Proposition 2.2. Applying the construction one finds the actions α, β as stated. We note that given these expressions, the required relations to construct $(\tilde{U}(R), A(R)^{\mathrm{op}})$ can also be verified directly in a manner similar to that of Theorem 3.2. The new part is to check consistency with the mixed l^+, l^- relations. The resulting double cross product $\tilde{U}(R)_\beta \bowtie_\alpha A(R)^{\mathrm{op}}$ has three matrices of generators, say $l^\pm$, v. These have the relations of $\tilde{U}(R)$ and $A(R)^{\mathrm{op}}$ and $l_2^\pm v_1 = R_{12}^{\pm -1} v_1 l_2^\pm R_{12}^\pm$ where $R^+ = R$, $R^- = \tau(R^{-1})$.

In a different line of development, it is easy to see that the construction of Theorem 3.2 can be iterated. Thus

Proposition 3.4. *Let $A^{(m)}(R) = A^{(m-1)}(R) \bowtie A(R)$ where $A^{(1)}(R) = A(R)$ and $(A^{(m-1)}(R), A(R))$ are matched as follows. Explicitly, $A^{(m-1)}(R)$ has $m-1$ matrix generators $\{u(i)\}_{i=1}^{m-1}$ and the actions are*

$$\alpha_{u(i)_1}(u_2) = R^{-1} u_2 R, \quad \beta_{u_2}(u(i)_1) = R^{-1} u(i)_1 R, \quad i = 1, \ldots, m-1.$$

Proof. By induction: suppose the relations in $A^{(m-1)}(R)$ are $R u(i)_1 u(j)_2 = u(j)_2 u(i)_1 R$ for all $i \leqq j$. This is then matched with $A(R)$ and the generators of

$A^{(m)}(R)$ are then $u(i) \equiv (u(i) \otimes 1)$ for $i = 1, \ldots, m-1$ and $u(m) \equiv (1 \otimes u)$, and have relations of the same form as in $A^{(m-1)}(R)$.

The relations in $A^{(m)}(R)$ can be written in the following suggestive form. Write

$$R(i) = \begin{cases} R & \text{for } i \leqq 0 \\ \tau(R^{-1}) & \text{for } i > 0 \end{cases}$$

for $i \in \mathbf{Z}$. Then $A^{(m)}(R)$ is the algebra with m matrix generators and relations

$$R(i-j)u(i)_1 u(j)_2 = u(j)_2 u(i)_1 R(i-j), \qquad \forall i, j = 1, \ldots, m.$$

The $R(i)$ obey

$$R(i-j)_{12} R(i)_{13} R(j)_{23} = R(j)_{23} R(i)_{13} R(i-j)_{12}$$

for all $i, j \in \mathbf{Z}$. This is a discrete version of the parametrized QYBE: the *parametrized QYBE* in physics is this equation for $R(\lambda)$, where $\lambda \in \mathbf{C}$ is a parameter (the spectral parameter). It is central to the method of quantum inverse scattering (for an introduction see, e.g., [M2]). Associated to each solution of it, there is a bialgebra $A(\{R(\lambda)\})$ from which integrable quantum chains may be constructed. It is defined by generators $\{u(\lambda)\}$ and relations like the discrete form above. This raises the intriguing possibility that physically interesting examples of $A(\{R(\lambda)\})$ could be understood as "continuously iterated double cross products" in continuum versions of Proposition 3.4 and its variants (for example, using $A(R)^{\mathrm{op}}$ or both $A(R)$ and $A(R)^{\mathrm{op}}$ etc.). Also, in the triangular case (i.e. if $R = \tau(R^{-1})$), one can prove a periodicity theorem to the effect that $A^{(4)}(R) \cong A^{(1)}(I_4 \otimes R)$. Here I_4 denotes the matrix in $M_2(k) \otimes M_2(k)$ with all entries 1.

4. Concluding remarks

We conclude with a brief outline of a further generalization of bicrossproducts, to include cocycles. They arise naturally in the context of a non-Abelian version of the extension theory of [Sing]. Briefly, an extension of Hopf algebras (H_1, H_2) is a Hopf algebra E and maps $H_2 \xrightarrow{i} E \xrightarrow{p} H_1$, such that E is isomorphic to $H_1 \otimes H_2$ as a right H_2-module and left H_1-comodule. Here the module and comodule structures on E are induced by pull back along i and pushout along p. $H_1 \otimes H_2$ is the trivial extension. In the case of an Abelian pair of Hopf algebras (H_1, H_2) (and in a graded connected context) [Sing] showed that (in our notation) $E \cong H_1\,{}^{\beta[\psi]}\!\bowtie_{\alpha[\chi]} H_2$ where α, β are as for a bicrossproduct, $\chi \in Z^2_\alpha(H_1, H_2)$ and $\psi \in Z^{2\beta}(H_1, H_2 \otimes H_2)$. Here Z_α and Z^β are the usual Hopf algebra cocycles coming from the inhomogeneous acyclic bar construction and its dual cobar construction respectively. (χ, ψ) are required to be compatible to obtain a Hopf algebra: [Sing] interpreted this as closure in a certain double complex, $(\chi, \psi) \in Z^{3\beta}_\alpha(H_1, H_2)$. Moreover, the class of E up to an obvious equivalence is classified by the class of $(\chi, \psi) \in H^{3\beta}_\alpha(H_1, H_2)$ [Sing].

These constructions have the following non-Abelian analog: $H_1\,{}^{\beta[\psi]}\!\bowtie_{\alpha[\chi]} H_2$. Here χ is a "2-cocycle" which, together with an "action" α, forms a right cocycle cross product system (as arising in other contexts in the works of [BCMD]). Explicitly, $\alpha: H_2 \otimes H_1 \to H_2: a \otimes h \mapsto a \triangleleft h$ respects the algebra structure on H_2 and obeys $(a \triangleleft h) \triangleleft g = \chi^{-1}(h_{(1)}, g_{(1)})(a \triangleleft h_{(2)}g_{(2)})\chi(h_{(3)}, g_{(3)})$, $a \triangleleft 1 = a$. $\chi: H_1 \otimes H_1 \to H_2$ obeys the "2-cocycle" condition

$$\chi(h_{(1)}g_{(1)}, f_{(1)})(\chi(h_{(2)}, g_{(2)}) \triangleleft f_{(2)}) = \chi(h, g_{(1)}f_{(1)})\chi(g_{(2)}, f_{(2)}),$$
$$\chi(1, h) = \chi(h, 1) = 1\varepsilon(h).$$

In this case there is a *cocycle cross product algebra* structure

$$(h \otimes a)(g \otimes b) = h_{(1)}g_{(1)} \otimes \chi(h_{(2)}, g_{(2)})(a \triangleleft g_{(3)})b.$$

Similarly $\psi: H_1 \to H_2 \otimes H_2$ and $\beta: H_1 \to H_2 \otimes H_1$ obey dual conditions to form a left cross coproduct system leading to a *cocycle cross coproduct coalgebra*. These two systems obey compatibility conditions (A), (B), (C) analogous to those in Proposition 1.1,

$$\varepsilon(a \triangleleft h) = \varepsilon(a)\varepsilon(h), \qquad \text{(A)}$$
$$\psi(h_{(1)})(\Delta a \triangleleft h_{(2)})\psi^{-1}(h_{(3)}) = (a_{(1)} \triangleleft h_{(1)})h_{(2)}{}^{(\bar 1)} \otimes a_{(2)} \triangleleft h_{(2)}{}^{(\bar 2)}$$

$$\beta(1) = 1 \otimes 1, \qquad \text{(B)}$$
$$(\chi^{-1}(h_{(1)}, g_{(1)}) \otimes 1)\beta(h_{(2)}g_{(2)})(\chi(h_{(3)}, g_{(3)}) \otimes 1) = (h^{(\bar 1)} \triangleleft g_{(1)})g_{(2)}{}^{(\bar 1)} \otimes h^{(\bar 2)}g_{(2)}{}^{(\bar 2)}$$

$$h_{(1)}{}^{(\bar 1)}(a \triangleleft h_{(2)}) \otimes h_{(1)}{}^{(\bar 2)} = (a \triangleleft h_{(1)})h_{(2)}{}^{(\bar 1)} \otimes h_{(2)}{}^{(\bar 2)} \qquad \text{(C)}$$

In addition (χ, ψ) now obey a compatibility condition

$$\psi(h_{(1)}g_{(1)})(\Delta\chi(h_{(2)}, g_{(2)}))\psi^{-1}(g_{(3)}) \qquad \text{(D)}$$
$$= \chi(h_{(1)}, g_{(1)})h_{(2)}{}^{(\bar 1)} \triangleleft g_{(2)}g_{(3)}{}^{(\bar 1)}(\psi(h_{(3)})^{(1)} \triangleleft g_{(4)})g_{(5)}{}^{(\bar 1)} \otimes \chi(h_{(2)}{}^{(\bar 2)}, g_{(3)}{}^{(\bar 2)})(\psi(h_{(3)})^{(2)} \triangleleft g_{(5)}{}^{(\bar 2)})$$

(along with subsidiary conditions

$$\varepsilon(\chi(h,g))=\varepsilon(h)\varepsilon(g),\ \psi(1)=1\otimes 1,\quad \chi(h_{(1)},Sh_{(2)})=\chi(Sh_{(1)},h_{(2)})=1\varepsilon(h),$$
$$(S\psi(h)^{(1)})\psi(h)^{(2)}=\psi(h)^{(1)}S\psi(h)^{(2)}=1\varepsilon(h)).$$

The resulting *cocycle bicrossproduct* $H_1\,{}^{\beta[\psi]}\!\bowtie_{\alpha[\chi]} H_2$ is an extension $H_2\to H_1\,{}^{\beta[\psi]}\!\bowtie_{\alpha[\chi]} H_2\to H_1$. Motivated by the extension theory of [Sing] we should therefore think of the conditions (D) etc. as a "non-Abelian 3-cocycle" condition, $(\chi,\psi)\in Z^{3\beta}_{\alpha}(H_1,H_2)$, and the equivalence classes of the corresponding extensions as defining a "non-Abelian cohomology" $H^{3\beta}_{\alpha}(H_1,H_2)$.

We note that the equations (A)–(D) simplify somewhat in the case that χ, ψ are central in their respective convolution algebras. (A)–(C) become the same as for a bicrossproduct $H_1\,{}^{\beta}\!\bowtie_{\alpha} H_2$ with the result that $H_1\,{}^{\beta[\psi]}\!\bowtie_{\alpha[\chi]} H_2$ can be viewed as a central extension of $H_1\,{}^{\beta}\!\bowtie_{\alpha} H_2$ by (χ,ψ). There is an analogous cocycle construction for double cross products. Clearly then, to find non-trivial examples of such cocycles and to understand them in terms of non-Abelian cohomology are two interesting directions for further work.

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